PLEASE ANSWER ALL QUESTIONS.
PLEASE EXPLAIN YOUR ANSWERS.

1. (a) Denote the normal-form game below by G. Solve G by iterated elimination of strictly dominated strategies. Explain briefly each step (1 sentence).

Player 2

Player 1		t_1	t_2	t_3
	s_1	$2,6$	$3,6$	$3,2$
	s_2	$1,4$	$4,4$	$0,5$
	s_3	$3,2$	$5,1$	$1,1$
	s_4	$4,4$	$2,1$	$4,0$

Solution: s_2 is dominated by s_3. After eliminating s_2, then t_3 is dominated by t_1. After eliminating t_3, then s_1 is dominated by s_3. After eliminating s_1, then t_2 is dominated by t_1. After eliminating t_2, then s_3 is dominated by s_4. Solution: (s_4, t_1).

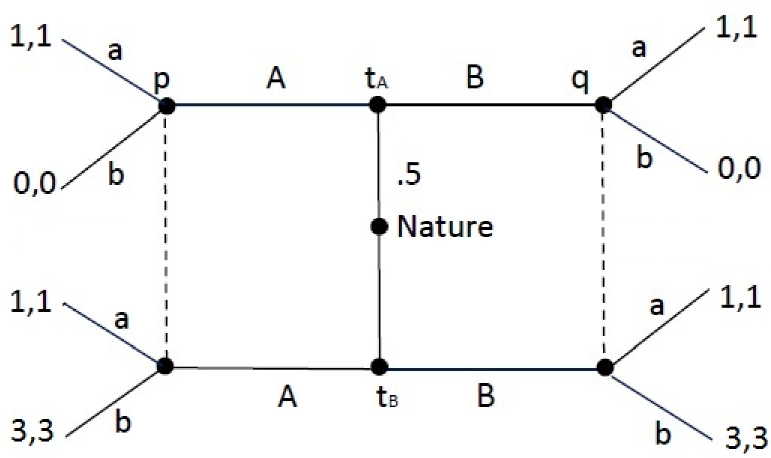
(b) Suppose we repeat G twice. Denote the resulting game by $G(2)$. Find the set of Subgame-perfect Nash Equilibria of $G(2)$. Be careful to write out the equilibrium strategies.

Solution: Since there is a unique outcome of the iterated elimination of strictly dominated strategies, this is the unique NE. Hence, it must be played in every subgame of the finitely repeated game. $SPNE = \{(\text{play } (s_4, t_1) \text{ in every subgame})\}$.

(c) How would we have to modify the payoffs in G to make it possible that there exists a Subgame-perfect Nash Equilibrium in which an action profile which is *not* a stage-game NE is played in one of the stages in $G(2)$? Explain this intuitively or give an example of a change in the payoffs.

Solution: There would have to be at least two stage-game NE.

2. Consider the game below, where sender observes nature's choice of t, and chooses the message A or B. Receiver does not observe t, but observes player 1's choice of message and chooses a or b.



(a) Is this a cheap talk game? Is it a game of coordination or conflict? Explain your answers.

Solution: This is a cheap talk game (payoffs depend only on type t and receiver's action, not on sender's message). It is a game of coordination, since both players want the same action to be taken in each state.

(b) Find a separating Perfect Bayesian Equilibrium.

Solution: Any of the following are separating PBE: $\{(AB, ab; p = 1, q = 0), (BA, ba; p = 0, q = 1)\}$.

(c) Find a pooling Perfect Bayesian Equilibrium in which the sender always sends the message A.

Solution: If sender always sends the message A, Bayes' Rule yields $p = 1/2$, and receiver will then take the action b (expected payoff $\frac{1}{2}(0) + \frac{1}{2}(3)$ versus $\frac{1}{2}(1) + \frac{1}{2}(1)$). Playing A will be optimal for sender if receiver also plays b after observing B (since otherwise, type t_A would deviate). The receiver will play b after observing B if

$$q(0) + (1 - q)(3) \geq q(1) + (1 - q)(1) \Leftrightarrow q \leq \frac{2}{3}.$$

Thus, the PBE is $(AA, bb; p = 1/2, q \leq 2/3)$.

(d) Compare the payoffs in the two equilibria you found in parts (b) and (c): does one equilibrium Pareto dominate the other?

Solution: The separating equilibrium always gives the highest possible payoff to both players, whereas the pooling equilibrium does not (when the type is t_A). Thus, the separating equilibrium Pareto dominates the pooling equilibrium.

(e) Do the equilibria satisfy SR6 (equilibrium domination)?

Solution: Yes. The separating equilibrium satisfies SR6 since there are no out-of-equilibrium beliefs. The pooling equilibrium because there is no equilibrium-dominated action for any of the two types.

3. Consider a *first-price sealed bid auction* with two bidders, who have valuations v_1 and v_2, respectively. For $i = 1, 2$, these values are distributed independently and uniformly with

$$v_i \sim u(2, 4).$$

Thus, the values are *private*.

Show that there is a symmetric Bayesian Nash Equilibrium in linear strategies: $b_i(v_i) = cv_i + d$, $i = 1, 2$. Find c and d.

Solution. Recall that $\mathbb{P}\left(\frac{b_i - d}{c} > v_j\right) = \frac{\frac{b_i - d}{c} - 2}{4 - 2} = \frac{b_i - d - 2c}{2c}$. We follow the procedure seen in the lecture. Assume that bidder j follows his proposed equilibrium strategy $b_j(v_j) = cv_j + d$. Then calculate the expected payoff to i from bidding b_i:

$$\begin{aligned}
\mathbb{E}[u_i(b_i, v_i)] &= \mathbb{P}(i \text{ wins}|b_i)(v_i - b_i) \\
&= \mathbb{P}(b_i > b_j(v_j))(v_i - b_i) \\
&= \mathbb{P}(b_i > cv_j + d)(v_i - b_i) \\
&= \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right)(v_i - b_i)
\end{aligned}$$

Thus

$$\mathbb{E}[u_i(b_i, v_i)] = \frac{b_i - d - 2c}{2c}(v_i - b_i).$$

Take the first-order condition

$$\frac{1}{2c}\left[(v_i - 2b_i) + (d + 2c)\right] = 0.$$

Easy to check SOC. Hence, best response is

$$b_i(v_i) = \frac{1}{2}v_i + \frac{1}{2}(d + 2c).$$

Therefore, $c^* = \frac{1}{2}$ and $d^* = \frac{1}{2}(d^* + 2c^*) = \frac{1}{2}(d^* + 1)$, which solves for $d^* = 1$. I.e. $b_i^*(v_i) = \frac{1}{2}v_i + 1$.

4. Consider the following version of Spence's education signaling model, where a firm is hiring a worker. The worker is characterized by his type θ, which measures his ability. There are two worker types: $\theta \in \{\theta_L, \theta_H\}$. Nature chooses the worker's type, with $\mathbb{P}(\theta = \theta_H) = p$ and $\mathbb{P}(\theta = \theta_L) = 1 - p$. The worker observes his own type, but the firm does not observe the worker's type.

   The worker can choose his level of education: $e \in \mathbb{R}^+$. The cost to him of acquiring this education is

   $$c_\theta(e) = 2 \cdot \frac{e^2}{\theta}.$$

   Education is observed by the firm, who then forms beliefs about the worker's type: $\mu(\theta|e)$. We assume that the marginal productivity of a worker is equal to his ability, and that the company is in competition such that it pays the expected marginal productivity:

   $$w(e) = \mathbb{E}(\theta|e).$$

   Thus, the payoff to a worker conditional on his type and education is

   $$u_\theta(e) = w(e) - c_\theta(e).$$

   Suppose for this exercise that $\theta_H = 6$ and $\theta_L = 2$.

   (a) Show that there is a *separating* pure strategy Perfect Bayesian Equilibrium where the low-ability worker chooses $e_L^* = 0$ and the high-ability worker chooses $e_H^* = 2$. You can use the off-equilibrium-path beliefs $\mu(\theta_H|e) = 0$ if $e \notin \{e_L^*, e_H^*\}$.

   Solution: Given the beliefs, $w(e) = 2$ for all $e \neq 2$ and $w(2) = 6$. Thus, all $e \notin \{0, 2\}$ are strictly dominated by $e = 0$, and will never be chosen in equilibrium. We therefore restrict attention to $e \in \{0, 2\}$. Notice $u_L(0) = 2$ and $u_L(2) = 6 - 2\frac{2^2}{2} = 2$. Thus, the low-ability type is indifferent between $e = 0$ and $e = 2$. On the other hand, $u_H(2) = 6 - 2\frac{2^2}{6} = \frac{14}{3}$ and $u_H(0) = 2$, so the high-ability type strictly prefers $e = 2$ over $e = 0$. Thus, $e_L^* = 0$ and $e_H^* = 2$ together with the specified beliefs is a PBE.

   (b) Find a *pooling* pure strategy Perfect Bayesian Equilibrium, where both worker types choose the same education level $e_p > 0$. What is the value of e_p in this pooling equilibrium? Give some intuition as to whether or not this pooling equilibrium is unique.

   Solution: On the equilibrium path $\mu(\theta_H|e_p) = p$. The off-the-equilibrium path beliefs that give the least incentives to deviate are $\mu(\theta_H|e) = 0$ if $e \neq e_p$. Thus, $w(e_p) = 6p + (1-p)2 = 2 + 4p$ and $w(e) = 2$ otherwise. Again, all $e \notin \{0, e_p\}$ are strictly dominated by $e = 0$, and will never be chosen in equilibrium. We therefore restrict attention to $e \in \{0, e_p\}$. First, we find the e_p such that the low-ability worker prefers $e = e_p$ to $e = 0$: $u_L(e_p) \geq u_L(0) \Leftrightarrow 2 + 4p - 2\frac{e_p^2}{2} \geq 2 \Rightarrow e_p \leq 2\sqrt{p}$. Similarly, we find the e_p such that the high-ability worker prefers $e = e_p$ to $e = 0$: $u_H(e_p) \geq u_H(0) \Leftrightarrow 2 + 4p - 2\frac{e_p^2}{6} \geq 2 \Rightarrow e_p \leq 2\sqrt{3p}$. Thus, $e_L^* = e_H^* \leq 2\sqrt{p}$ together with the specified beliefs is a PBE.

   As shown above, the pooling equilibrium is clearly not unique. Any level of pooling that is not too high can be an equilibrium. They key is that a deviator will be 'punished' by a lower wage, which will assure that all types stick to the equilibrium strategy.